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FREE GROUP REPRESENTATIONS FROM VECTOR-VALUED MULTIPLICATIVE FUNCTIONS, I*

BY

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ABSTRACT

Let Γ denote a noncommutative free group, and let Ω stand for its boundary. We construct a large class of unitary representations of Γ. This class contains many previously studied representations, and is closed under several natural operations. Each of the constructed representations is in fact a representation of $\Gamma \ltimes_\lambda C(\Omega)$. We prove here that each of them is irreducible as a representation of $\Gamma \ltimes_\lambda C(\Omega)$. Actually, as will be shown in further work, each of them is irreducible as a representation of Γ, or is the direct sum of exactly two irreducible, inequivalent Γ-representations.

1. Introduction

Let Γ be a finitely generated noncommutative free group. For brevity, we will say that a unitary representation $(\pi, \mathcal{H})$ of G is **tempered** if it is weakly contained in the regular representation. The following is a partial list of works in which specific irreducible tempered representations of Γ have been constructed and proved irreducible: Yoshizawa, 1951 [Yos], Figà-Talamanca and Picardello, 1982

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and 1983 [F-T–P1] and [F-T–P2], Angelini, 1989 [A], Figà-Talamanca Steger, 1994 [F-T–S], Kuhn Steger, 1996 [K–S1], Paschke, 2001 and 2002 [P1] and [P2]*. Constructing the representations is typically easy; proving them irreducible is typically hard. Similarly proving that two of the constructed representations are not equivalent is usually hard.

In this series of papers our objective is to construct a large class of irreducible tempered representations of Γ, to prove them irreducible, and to say exactly when two of them are equivalent. The class to be constructed includes all the irreducible tempered representations given in the papers listed above. Although our construction does depend on the choice of a set of generators, the class of representations obtained is independent of that choice. This class is closed under passage to (the irreducible components of) tensor products with finite dimensional representations and likewise under passage to (the irreducible components of) restrictions to finite index subgroups.

To get an example not covered by the present construction embed Γ in $SL(2,\mathbf{R})$ as a lattice subgroup and then restrict any principal series representation of the latter to Γ. The result is a tempered irreducible representation (see Cowling–Steger, 1991, [C–S]) not obtainable using the construction presented here. Contrariwise, if Γ is embedded as a lattice in $PGL(2,\mathbf{Q}_p)$, the corresponding restrictions are obtainable. In the first case Γ is acting on the hyperbolic plane; in the second case on a tree. As far as we know, the construction presented here is sufficient to get at all specific, irreducible, tempered representations hitherto constructed using "tree" methods.

2. The boundary and boundary representations

Fix a set, A_+, of free generators for Γ. Let A_- be the set of inverses of these generators, and let $A = A_+ \amalg A_-$. Throughout, when we use a, b, c, and d for elements of Γ, it is implied that they are elements of A. There is a unique **reduced word** for every $x \in \Gamma$:

$$x = a_1 a_2 \ldots a_n \quad \text{where for all } j,\ a_j \in A \text{ and } a_j a_{j+1} \neq e.$$

* Other relevant irreducible representations have been constructed by a school centred in Wrocław including T. Pytlik, R. Swarc, M. Bożejko, J. Wysoczański, W. Młotkowski and W. Hebisch. This school has not had any special interest in temperedness. Indeed their constructions often yield representations which are not necessarily unitary, let alone tempered. Also, they have often worked with free products of finite groups, rather than with the free group. Nonetheless, some particular cases of their constructions, sometimes with slight modifications, could also be included in the above list.

If there are n factors in the reduced word for x, we write $|x| = n$.

The **Cayley graph** of Γ has as vertices the elements of Γ and as undirected edges the couples $\{x, xa\}$ for $x \in \Gamma$, $a \in A$. This is a tree with $\#(A)$ edges attached to each vertex. The action of Γ on itself by left-translations preserves the tree structure. Define the distance between two vertices of the tree as the number of edges in the path joining them. This gives $d(e,x) = |x|$, $d(x,y) = |x^{-1}y|$.

Say that a sequence $(x_0, x_1, \ldots)$ of vertices in the tree is a **semi-infinite geodesic** if for all j, x_{j+1} is adjacent to x_j and $x_{j+2} \neq x_j$. Define an equivalence relation on the set of semi-infinite geodesics, saying that two geodesics are equivalent if, up to a shift in indices, they differ only in finitely many places; i.e., say that $(x_j)_j \sim (y_j)_j$, if there exist $J_1, J_2 \geq 0$ such that $x_{j+J_1} = y_{j+J_2}$ for all $j \geq 0$. The set of equivalence classes is called the **boundary of the tree**, or the **boundary of** Γ, and is denoted Ω.

The left action of Γ on the tree gives rise to an action of Γ on Ω. For any $\omega \in \Omega$, there is a unique semi-infinite geodesic starting at the vertex e and representing ω. The vertices of this geodesic will have the form

$$(e, a_1, a_1a_2, a_1a_2a_3, \ldots) \quad \text{where for all } j,\ a_j \in A \text{ and } a_ja_{j+1} \neq e.$$

We identify this ω with the **infinite reduced word** $a_1a_2a_3\ldots$. One may check that the left action of Γ on Ω corresponds to the obvious multiplication of finite and infinite reduced words. Consequently, we denote by $x\omega$ the action of $x \in \Gamma$ on $\omega \in \Omega$.

For $x \in \Gamma$ let

$$\Gamma(x) = \{y \in \Gamma;\ \text{the reduced word for } y \text{ starts with } x\},$$
$$\Omega(x) = \{\omega \in \Omega;\ \text{the reduced word for } \omega \text{ starts with } x\}.$$

We topologize Ω using the sets $\Omega(x)$ as a basis. The sets $\Omega(x)$ are then both closed and open in Ω, and Ω is a compact, Hausdorff, perfect, separable, totally disconnected space — in other words, homeomorphic to the Cantor set. Likewise, we topologize $\Gamma \amalg \Omega$ using the sets $\Gamma(x) \amalg \Omega(x)$ and the singletons $\{x\}$ as a basis. This makes $\Gamma \amalg \Omega$ a compact space. If $(x_j)_j$ is a semi-infinite geodesic representing $\omega \in \Omega$, then with this topology we have $\omega = \lim_j x_j$. One has $\Omega(x) = \overline{\Gamma(x)} \cap \Omega$. On both Ω and $\Gamma \amalg \Omega$, the action of Γ is by homeomorphisms.

For any directed edge (x, xa) of the tree define

$$\Gamma(x, xa) = \{y \in \Gamma;\ d(y, xa) < d(y, x)\}.$$

Upon removing that edge, the tree decomposes as $\Gamma = \Gamma(x, xa) \amalg \Gamma(xa, x)$. Suppose that $|xa| = |x| + 1$, that is, suppose that a is the last letter in the reduced word for xa. Then $\Gamma(x, xa) = \Gamma(xa)$ (while if $|xa| = |x| - 1$, then $\Gamma(x, xa) = \Gamma \sim \Gamma(x)$). If $|y| < |xa| = |x| + 1$, then also $|yxa| = |yx| + 1$, and so

$$y\Gamma(xa) = y\Gamma(x, xa) = \Gamma(yx, yxa) = \Gamma(yxa),$$
$$y\Omega(xa) = y(\overline{\Gamma(xa)} \cap \Omega) = \overline{y\Gamma(xa)} \cap \Omega = \overline{\Gamma(yxa)} \cap \Omega = \Omega(yxa).$$

Considering, by contrast, the case $y = (xa)^{-1}$, one finds

$$(xa)^{-1}\Gamma(xa) = (xa)^{-1}\Gamma(x, xa) = \Gamma(a^{-1}, e) = \Gamma \sim \Gamma(a^{-1}).$$

Let $C(\Omega)$ be the commutative C^*-algebra of continuous complex valued functions on Ω, under pointwise operations. Let $\lambda\colon \Gamma \to \mathrm{Aut}(C(\Omega))$ be the action by translations: $(\lambda(x)F)(\omega) = F(x^{-1}\omega)$.

Definition 2.1: A **boundary representation** is a triple $(\pi_\Gamma, \pi_\Omega, \mathcal{H})$ where

- $\pi_\Omega\colon C(\Omega) \to \mathcal{L}(\mathcal{H})$ is a $*$-representation of $C(\Omega)$,
- $\pi_\Gamma\colon \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ, and
- for any $x \in \Gamma$ and $F \in C(\Omega)$

$$\pi_\Gamma(x)\pi_\Omega(F)\pi_\Gamma(x^{-1}) = \pi_\Omega(\lambda(x)F).$$

Normally we will suppress the subscripts and write π for both π_Γ and π_Ω. The reader who is familiar with crossed-product C^*-algebras will recognize that a boundary representation is just a representation of $\Gamma \ltimes_\lambda C(\Omega)$. The reader without this familiarity does not need to acquire it; it is enough to understand the above definition of boundary representation.

Here is an almost universal construction of a boundary representation. Start with the following data:

- a positive quasi-invariant Borel measure ν on Ω,
- a Hilbert space $\mathcal{V}$ (which may be finite dimensional) and
- a **cocycle** $A\colon \Gamma \times \Omega \to \mathcal{U}(\mathcal{V})$ satisfying

$$A(xy, \omega) = A(x, \omega)A(y, x^{-1}\omega)$$

  and ν-measurable with respect to ω.

Then define

- $\mathcal{H} = L^2(\Omega, d\nu, \mathcal{V})$,
- $(\pi(F)v)(\omega) = F(\omega)v(\omega)$ for $F \in C(\Omega)$ and $v \in \mathcal{H}$, and
- $(\pi(x)v)(\omega) = (\frac{d\nu(x^{-1}\omega)}{d\nu(\omega)})^{1/2}A(x, \omega)v(x^{-1}\omega)$ for $x \in \Gamma$ and $v \in \mathcal{H}$.

Any boundary representation may be obtained as a direct sum of such representations, where one may take the dimensions of the various $\mathcal{V}$ to be distinct, and the various measures ν to be mutually singular.

The following definition and observations are completely standard.

Definition 2.2: If π is a boundary representation acting on $\mathcal{H}$, then a **subrepresentation** of π is a closed subspace of $\mathcal{H}$ invariant under both $\pi(\Gamma)$ and $\pi(C(\Omega))$, with the restricted actions of Γ and $C(\Omega)$. If $\mathcal{H} \neq 0$ and if the only subrepresentations of π are 0 and $\mathcal{H}$ itself, one says that π is an **irreducible** boundary representation.

If $\pi^\sharp$ acting on $\mathcal{H}^\sharp$ is another boundary representation, then a map $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}^\sharp$ satisfying $\pi^\sharp(x)\mathcal{J} = \mathcal{J}\pi(x)$ for $x \in \Gamma$ and $\pi^\sharp(F)\mathcal{J} = \mathcal{J}\pi(F)$ for $F \in C(\Omega)$ is called a **map of boundary representations** or an **intertwiner** from π to $\pi^\sharp$. Two boundary representations π and $\pi^\sharp$ are called **equivalent** if there exists a unitary intertwiner between them.

Observation 2.3: If $\mathcal{J}$ is a map of boundary representations, so is $\mathcal{J}^*$. Likewise, sums and products of maps of boundary representations are maps of boundary representations. If $\mathcal{H}_1 \subseteq \mathcal{H}$ is a subrepresentation of π, so is its orthogonal complement, $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$, and the orthogonal projection $P\colon \mathcal{H} \to \mathcal{H}$ projecting onto $\mathcal{H}_1$ is a map of boundary representations. Between two inequivalent, irreducible boundary representations there are no nonzero intertwiners.

FACT 2.4: When a boundary representation (i.e., a representation of $\Gamma \ltimes_\lambda C(\Omega)$) is considered as a representation of Γ (i.e., is restricted to $C^*(\Gamma)$) it is always a tempered representation (i.e., a representation of $C^*_{\mathrm{red}}(\Gamma)$).

This follows from the general considerations in Quigg–Spielberg, 1992 [Q–S]. A two-page proof specifically for the case at hand may be found in Section 2 of Kuhn–Steger, 1996 [K–S2].

One now sees how easy it is to construct tempered representations of Γ. As above, choose any quasi-invariant measure ν on Ω, and any Hilbert space $\mathcal{V}$, for example $\mathcal{V} = \mathbf{C}$. Then choose, for each generator $a \in A_+$, an arbitrary measurable map $A(a,\cdot)\colon \Omega \to \mathcal{U}(\mathcal{V})$. This gives a unitary action of each generator a on $\mathcal{H} = L^2(\Omega, d\nu, \mathcal{V})$. By the universal property of free groups, this extends to an action of Γ on $\mathcal{H}$. One can check that the extended action is of the form given in the above construction of boundary representations. It follows from Fact 2.4 that any representation so constructed is tempered.

Both experience and intuition suggest that "most" such choices will lead to representations which are irreducible as well as tempered. So any claim of universality which we make for the construction presented in this series of papers is to be taken with many grains of salt. We claim to have covered many (or even most) of the specific irreducible tempered representations occurring in the literature, but we have not even begun to touch the sea of possibilities.

Indeed, one knows that the situation is hopeless. Irreducible tempered representations of Γ are none other than irreducible representations of $C^*_{\mathrm{red}}(\Gamma)$. Because Γ is discrete, that algebra is traceable. According to Powers, 1975 [Pow] it is also simple. Such an algebra is "Type II", and so one knows that its unitary dual cannot be parameterized by any standard measure space. This rules out any reasonably concrete parametrization: no one will ever be able to "list" all possible tempered unitary representations of Γ.

In this paper, the first of a series, we construct a family of boundary representations. Section 3 explains what a **matrix system** (or just **system**) (V_a, H_{ba}) is, when such a system is irreducible, and when two matrix systems are equivalent. Next it explains what a **system with inner products** (V_a, H_{ba}, B_a) is, and how to construct a boundary representation π out of a system with inner products.

When it is possible to extend an irreducible matrix system (V_a, H_{ba}) to a system with inner products (V_a, H_{ba}, B_a), we say that the system is **normalized**. Section 4 shows that an irreducible system is normalized if and only if it satisfies a single metric condition, $\rho = 1$. Only normalized systems are of interest here. If the irreducible matrix system is normalized, it is shown that the extension to a system with inner products is essentially unique. In particular, the (equivalence class of the) constructed representation π is completely determined by the irreducible system (V_a, H_{ba}).

Section 5 demonstrates the irreducibility of the boundary representation π constructed from an irreducible matrix system (V_a, H_{ba}), and the inequivalence of boundary representations π and $\pi^\sharp$ constructed from inequivalent irreducible systems. Also in Section 5 is the proof of a matrix theorem (Corollary 5.4) which is important in the later development. Section 6 explains how several important examples of Γ-representations can be obtained with the present construction.

The succeeding papers of the series will consider the representations constructed here as representations of Γ rather than as boundary representations (representations of $C^*_{\mathrm{red}}(\Gamma)$ rather than representations of $\Gamma \ltimes_\lambda C(\Omega)$). In other words, those papers will concentrate on π_Γ to the exclusion of π_Ω. Nonetheless, the results of this first paper are essential ingredients in what follows.

Here is a brief and incomplete synopsis of the further results. Generically, the representation π_Γ constructed here from an irreducible matrix system is irreducible, but in certain so-called **odd** cases it decomposes as the direct sum of two inequivalent irreducible Γ-representations. Generically, representations π_Γ and $\pi_\Gamma^\sharp$ constructed from inequivalent irreducible systems are inequivalent, but it may happen that exactly two (equivalence classes of) irreducible systems give rise to the same Γ-representation. This is the case of **duplicity**. We work out exactly which pairs of irreducible systems are duplicitous and exactly which irreducible systems are odd. The matrix coefficients of some Γ-representations are, in a certain precise sense, smaller than those of others. The smallest matrix coefficients occur exactly in the odd and duplicitous cases.

3. Multiplicative functions and representations

Definition 3.1: A **matrix system** or simply **system** (V_a, H_{ba}) is obtained by choosing

- a finite-dimensional vector space V_a for each $a \in A$, and
- a linear map $H_{ba}\colon V_a \to V_b$ for each pair $a, b \in A$, where $H_{ba} = 0$ whenever $ab = e$.

Definition 3.2: A tuple of linear subspaces $W_a \subseteq V_a$ is called an **invariant subsystem** of (V_a, H_{ba}) if

$$H_{ba} W_a \subseteq W_b \quad \text{for all } a, b.$$

The system (V_a, H_{ba}) is called **irreducible** if it is nonzero and if it admits no invariant subsystems except for itself and the zero subsystem.

Definition 3.3: A map from the system (V_a, H_{ba}) to the system $(V_a^\sharp, H_{ba}^\sharp)$ is a tuple (J_a) where $J_a\colon V_a \to V_a^\sharp$ is a linear map and

$$H_{ab}^\sharp J_b = J_a H_{ab}. \tag{3.3}$$

The tuple (J_a) is called an **equivalence** if each J_a is a bijection. Two systems are called **equivalent** if there is an equivalence between them.

Remark 3.4: A map (J_a) between irreducible systems (H_{ba}, V_a) and $(H_{ba}^\sharp, V_a^\sharp)$ is either 0 or an equivalence. This is because the kernels (respectively, the images) of the maps J_a constitute an invariant subsystem.

Given an irreducible system (V_a, H_{ba}) which satisfies one additional condition $(\rho = 1)$ we are going to construct a boundary representation. As the first step one defines $\mathcal{H}^\infty$, the space of **(vector-valued) multiplicative functions**. $\mathcal{H}^\infty$

will be a dense subspace of the eventual representation space $\mathcal{H}$. A **(vector-valued) multiplicative function** is a function $f\colon \Gamma \to \coprod_{a\in A} V_a$ satisfying the following condition: there exists $N = N(f)$ such that for every $x \in \Gamma$, with $|x| \geq N$

$$\begin{aligned} f(xa) &\in V_a && \text{if } |xa| = |x| + 1, \\ f(xab) &= H_{ba} f(xa) && \text{if } |xab| = |x| + 2. \end{aligned} \tag{3.2}$$

Note that if f satisfies (3.2) for some $N = N_0$, it also satisfies (3.2) for all $N \geq N_0$. We declare that two multiplicative functions f and g are equivalent if $f(x) = g(x)$ for all but finitely many elements of Γ. $\mathcal{H}^\infty$ will be the quotient of the space of multiplicative functions with respect to this equivalence relation.

The vector space structure on $\mathcal{H}^\infty$ is given by pointwise addition and pointwise multiplication by scalars. When calculating $f_1 + f_2$, one uses the fact that $f_1(x)$ and $f_2(x)$ belong to the same space V_a for all but finitely many x. For those finitely many x where there is a discrepancy, one may choose any arbitrary value for $(f_1 + f_2)(x)$.

Next one wishes to introduce an inner product on $\mathcal{H}^\infty$. For this, one needs for each $a \in A$ a positive definite sesquilinear form B_a on $V_a \times V_a$. The desired definition of the inner product on $\mathcal{H}^\infty$ is

$$\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_{\mathcal{H}} = \sum_{|x|=N} \sum_{\substack{a \\ |xa|=|x|+1}} B_a(f_1(xa), f_2(xa)) \tag{3.3}$$

where N is big enough so that both f_i satisfy (3.2). For the definition to be independent of N it is necessary that for $a \in A$ and $v_a \in V_a$ one has $B_a(v_a, v_a) = \sum_b B_b(H_{ba} v_a, H_{ba} v_a)$.

Definition 3.5: The triple (V_a, H_{ba}, B_a) is a **system with inner products** if (V_a, H_{ba}) is a matrix system, if B_a is a positive definite sesquilinear form on V_a for each $a \in A$, and if for $a \in A$ and $v_a \in V_a$ one has

$$B_a(v_a, v_a) = \sum_b B_b(H_{ba} v_a, H_{ba} v_a). \tag{3.4}$$

In Section 4 below we discuss the possibility of extending an irreducible matrix system (V_a, H_{ba}) to a system with inner products. To summarize, the existence of a tuple (B_a) compatible with (V_a, H_{ba}) depends on an extra hypothesis ($\rho = 1$) while the uniqueness of (B_a) holds up to multiplication by a positive constant. At the moment we shall simply assume that such a tuple exists and proceed with the construction of the representation.

PROPOSITION 3.6: *Given (3.4), the inner product $\langle\cdot,\cdot\rangle$ of (3.3) is well defined.*

Proof: Assume that f_1 and f_2 are multiplicative. Choose N big enough so that they both satisfy (3.2). For $N' > N$, we need to show that the two definitions for $\langle f_1, f_2\rangle$ match.

Fix a vertex xa with $|x| = N$, $|xa| = N+1$. By (3.4) and (3.2) we have

$$B_a(f(xa), f(xa)) = \sum_b B_b(H_{ba}f(xa), H_{ba}f(xa)) = \sum_b B_b(f(xab), f(xab)).$$

Adding up over all a such that $|xa| = |x|+1$ and over all x of length N, we get

$$\begin{aligned}\langle f_1, f_2\rangle &= \sum_{|x|=N} \sum_{\substack{a\\ |xa|=N+1}} \sum_{\substack{b\\ |xab|=|x|+2}} B_b(f_1(xab), f_2(xab))\\ &= \sum_{|y|=N+1} \sum_{\substack{b\\ |yb|=|y|+1}} B_b(f_1(yb), f_2(yb)).\end{aligned}$$

Hence(3.3) gives the same result if one uses $N+1$ in place of N. Simple induction finishes the proof. ∎

It is clear that $\langle\cdot,\cdot\rangle$ is a positive sesquilinear form whose null space consists of the compactly supported functions. This fits with our definition of equivalence of multiplicative functions. We shall denote by $\mathcal{H}$ the completion of $\mathcal{H}^\infty$ with respect to the norm induced by $\langle\cdot,\cdot\rangle$.

For $v_a \in V_a$ define $\mu[x, xa, v_a]\colon \Gamma \to \coprod_{a\in A} V_a$ according to

- $\mu[x, xa, v_a](y) = 0$ for $y \notin \Gamma(x, xa)$,
- $\mu[x, xa, v_a](xa) = v_a$,
- $\mu[x, xa, v_a](ybc) = H_{cb}\mu[x, xa, v_a](yb)$ if $yb, ybc \in \Gamma(x, xa)$ with $d(ybc, x) = d(y, x) + 2$.

Thus if $ab_1 \dots b_n$ is reduced

$$\mu[x, xa, v_a](xab_1 \dots b_n) = H_{b_n b_{n-1}} \dots H_{b_2 b_1} H_{b_1 a} v_a$$

In this definition we do *not* require that $|xa| = |x| + 1$. So, for example, if $x = a_1 \dots a_n$ is the reduced word for x and $a = {a_n}^{-1}$, then $e \in \Gamma(x, xa)$ and $\mu[x, x{a_n}^{-1}, v_{{a_n}^{-1}}](e) = H_{{a_1}^{-1}{a_2}^{-1}} \dots H_{{a_{n-1}}^{-1}{a_n}^{-1}} v_{{a_n}^{-1}}$. On the other hand, if $|xa| = |x|+1$, then $\mu[x, xa, v_a]$ satisfies (3.2) for $N = |x|$, and so is multiplicative. Note that $y\Gamma(x, xa) = \Gamma(yx, yxa)$ and that $\mu[x, xa, v_a](y^{-1}\cdot) = \mu[yx, yxa, v_a](\cdot)$

Fix $f \in \mathcal{H}^\infty$. Then for N large enough, and modulo the equivalence relation,

$$f = \sum_{|x|=N} \sum_{a;|xa|=N+1} \mu[x, xa, f(xa)] \tag{3.5}$$

and so

$$f(y^{-1}\cdot) = \sum_{|x|=N} \sum_{a;|xa|=N+1} \mu[yx, yxa, f(xa)](\cdot) \tag{3.6}$$

Choose $N > |y|$. Then one always has $|yxa| = |yx| + 1$, hence each of the $\mu[yx, yxa, f(xa)]$ is multiplicative, hence $f(y^{-1}\cdot)$ is multiplicative. Define the representation π_Γ of Γ on $\mathcal{H}^\infty$ by setting

$$(\pi_\Gamma(y)f)(x) = f(y^{-1}x).$$

Normally, we will just write $\pi(y)$ in place of $\pi_\Gamma(y)$.

We now show that translation preserves the inner product. For $|xa| = |x|+1$, direct application of (3.3) gives $\langle \mu[x, xa, v_a], \mu[x, xa, v_a'] \rangle = B_a(v_a, v_a')$. The supports, $\Gamma(x, xa)$, of the terms in (3.5) are all disjoint, and so the supports $y\Gamma(x, xa) = \Gamma(yx, yxa)$ of the terms in (3.6) are all disjoint. Using the fact that multiplicative functions of disjoint support are always orthogonal, and choosing $N > |y|$ big enough,

$$\begin{aligned} &\langle \pi(y)f_1, \pi(y)f_2 \rangle \\ &= \langle \sum_{|x|=N} \sum_{a;|xa|=N+1} \mu[yx, yxa, f_1(xa)], \sum_{|x|=N} \sum_{a;|xa|=N+1} \mu[yx, yxa, f_2(xa)] \rangle \\ &= \sum_{|x|=N} \sum_{a;|xa|=N+1} B_a(f_1(xa), f_2(xa)) = \langle f_1, f_2 \rangle. \end{aligned}$$

Using this, extend $\pi(y)$ to all of $\mathcal{H}$ by continuity.

It now follows, even in the case $|xa| = |x|-1$, that $\mu[x, xa, v_a] = \pi(x)\mu[e, a, v_a] \in \pi(x)\mathcal{H}^\infty = \mathcal{H}^\infty$ and that

$$\langle \mu[x, xa, v_a], \mu[x, xa, v_a'] \rangle = \langle \pi(x)\mu[e, a, v_a], \pi(x)\mu[e, a, v_a'] \rangle = B_a(v_a, v_a').$$

Next we define the algebra homomorphism $\pi_\Omega \colon C(\Omega) \to \mathcal{B}(\mathcal{H})$, writing, as usual, π for π_Ω. In $C(\Omega)$ let $\mathbf{1}$ denote the function identically one, and let $\mathbf{1}_{\Omega(x)}$ denote the characteristic function of $\Omega(x)$. The subalgebra spanned by the functions $\{\mathbf{1}_{\Omega(x)}\}_{x\in\Gamma}$ is denoted by $C^\infty(\Omega)$ and consists of locally constant functions. It is a dense $*$-subalgebra of $C(\Omega)$. In order to define a continuous action of $C(\Omega)$ on $\mathcal{H}$ it is sufficient to define it for $C^\infty(\Omega)$. Denote by $\mathbf{1}_{\Gamma(x)}$ the characteristic function of the set $\Gamma(x)$. Define $\pi(\mathbf{1}_{\Omega(x)}) \colon \mathcal{H}^\infty \to \mathcal{H}^\infty$ by

$$(\pi(\mathbf{1}_{\Omega(x)})f)(y) = (\mathbf{1}_{\Gamma(x)}f)(y) = \begin{cases} f(y) & \text{if } y \in \Gamma(x), \\ 0 & \text{otherwise.} \end{cases}$$

Observation 3.7:

(1) $\pi(\mathbf{1}_{\Omega(x)}) = \sum_{a;|xa|=|x|+1} \pi(\mathbf{1}_{\Omega(xa)})$,
(2) $\pi(\mathbf{1}_{\Omega(x)}\mathbf{1}_{\Omega(y)}) = \pi(\mathbf{1}_{\Omega(x)})\pi(\mathbf{1}_{\Omega(y)})$
(3) $\pi(\mathbf{1}_{\Omega(x)})$ is a self-adjoint projection.

It follows from (3) that $\| \pi(\mathbf{1}_{\Omega(x)})\| \leq 1$, so we may extend $\pi(\mathbf{1}_{\Omega(x)})$ to a map on $\mathcal{H}$ via continuity. Any $F \in C^\infty(\Omega)$ can be written as $\sum_{|x|=N} F_x \mathbf{1}_{\Omega(x)}$ for big enough N. Define

$$\pi(F) = \pi\left(\sum_{|x|=N} F_x \mathbf{1}_{\Omega(x)}\right) = \sum_{|x|=N} F_x \pi(\mathbf{1}_\Omega(x)). \tag{3.7}$$

To see that $\pi(F)$ is well-defined, independent of N, use (1). To see that π is an algebra homomorphism, use (2), and to see it is a $*$-homomorphism use (3). From (2) and (3) it follows that $\{\pi(\mathbf{1}_{\Omega(x)})\}_{|x|=N}$ is a collection of orthogonal projections. Thus $\| \pi(F)\| \leq \| F\|_{C(\Omega)}$. Using this, extend π from $C^\infty(\Omega)$ to $C(\Omega)$ by continuity.

To see that π_Γ and π_Ω constitute a boundary representation (that is a representation of $\Gamma \ltimes_\lambda C(\Omega)$) we must show

$$\pi(x)\pi(F)\pi(x^{-1}) = \pi(\lambda(x)F)$$

for $x \in \Gamma$ and $F \in C(\Omega)$. By linearity and continuity it is enough to check this for $F = \mathbf{1}_{\Omega(y)}$. Decomposing F, if necessary, we may assume $|y| > |x|$, hence $x\Gamma(y) = \Gamma(xy)$, $x\Omega(y) = \Omega(xy)$, and $\lambda(x)\mathbf{1}_{\Omega(y)} = \mathbf{1}_{\Omega(xy)}$. For $f \in \mathcal{H}^\infty$

$$\begin{aligned}(\pi(x)\pi(\mathbf{1}_{\Omega(y)})\pi(x^{-1}))f(z) &= (\pi(\mathbf{1}_{\Omega(y)})\pi(x^{-1})f)(x^{-1}z)\\ = \mathbf{1}_{\Gamma(y)}(x^{-1}z)(\pi(x^{-1})f)(x^{-1}z) &= \mathbf{1}_{\Gamma(xy)}(z)f(z) = (\pi(\mathbf{1}_{\Omega(xy)})f)(z).\end{aligned}$$

According to Fact 2.4 we now know that π_Γ is tempered.

Remark 3.8: Recall that our irreducible system is not allowed to be zero. So for at least one $c \in A$, there exists nonzero $v_c \in V_c$. Fix $y \in \Gamma$. Then there exist $x, xc \in \Gamma(y)$ with $|xc| = |x| + 1$. Then $\pi(\mathbf{1}_{\Omega(y)})\mu[x, xc, v_c] = \mu[x, xc, v_c]$. We conclude

- $\| \pi(\mathbf{1}_{\Omega(y)})\| = 1$ for all y, hence
- $\| \pi(F)\| = \| F\|_{C(\Omega)}$ for all $F \in C(\Omega)$.

This also follows from the well-known fact that $\Gamma \ltimes_\lambda C(\Omega)$ is a simple C^*-algebra.

4. Existence of the system $(B_a)_{a\in A}$

We shall apply a generalized version of the Perron–Frobenius Theorem due to J. S. Vandergraft to guarantee the existence of a system of $(B_a)_{a\in A}$ satisfying (3.4). Here are some definitions needed. Let $\mathcal{S}$ be any *real* vector space.

Definition 4.1: A **solid cone** $\mathcal{K}$ is a subset $\mathcal{K}$ of $\mathcal{S}$ which satisfies the following:

- $\mathcal{K}$ is closed with nonempty interior,
- $\alpha\mathcal{K} = \mathcal{K}$ for all positive α,
- $\mathcal{K} + \mathcal{K} = \mathcal{K}$, and
- $\mathcal{K} \cap (-\mathcal{K}) = 0$.

The following is parts (i) and (iii) of Theorem 3.1 from Vandergraft, 1968 [Van]:

THEOREM 4.2: *Assume that $\mathcal{K}$ is a solid cone in some real finite dimensional space. Suppose that $\mathcal{T}$ is a linear operator which leaves $\mathcal{K}$ invariant. Then the spectral radius $\rho = \rho(\mathcal{T})$ of $\mathcal{T}$ is an eigenvalue and there exists an eigenvector corresponding to ρ which lies in $\mathcal{K}$.*

Definition 4.3: We shall refer to $\rho(\mathcal{T})$ as the **Perron–Frobenius** eigenvalue of $\mathcal{T}$.

The following discussion applies to any system (V_a, H_{ba}). For each $a \in A$, let S_a be the real vector space of symmetric sesquilinear forms on $V_a \times V_a$. Let $\mathcal{S} = \bigoplus_{a\in A} S_a$. Say that a tuple $(B_a) \in \mathcal{S}$ is **positive definite** (**positive semidefinite**) if each of its components is positive definite (positive semidefinite). Let $\mathcal{K} \subseteq \mathcal{S}$ denote the subset of positive semidefinite tuples, and observe that $\mathcal{K}$ is a solid cone according to Definition 4.1. Define $\mathcal{T}\colon \mathcal{S} \to \mathcal{S}$ by the rule

$$(\mathcal{T}C)_a(v_a, v_a) = \sum_b C_b(H_{ba}v_a, H_{ba}v_a). \tag{4.1}$$

Observe that $\mathcal{T}$ acts linearly on $\mathcal{S}$ and that $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$. Vandergraft's Theorem ensures that there exists a tuple $(B_a)_a$ of positive semidefinite forms on each V_a such that

$$(\mathcal{T}B)_a(v_a, w_a) = \sum_b B_b(H_{ba}v_a, H_{ba}w_a) = \rho B_a(v_a, w_a). \tag{4.2}$$

Definition 4.4: A system (V_a, H_{ba}) is said to be **normalized** if the Perron–Frobenius eigenvalue ρ of $\mathcal{T}$ is one.

When a system is normalized, the Perron–Frobenius eigentuple $(B_a)_a$ satisfies equation (3.4). Our next goal is to show that when a normalized system is *irreducible*, then the eigentuple $(B_a)_a$ is strictly positive definite and unique up to scalars.

LEMMA 4.5: *Let (V_a, H_{ba}) be a an irreducible matrix system. Suppose $P = (P_a)_a \in \mathcal{K}$ is a nonzero tuple of positive semidefinite sesquilinear forms satisfying $\mathcal{T}P = \lambda P$ for some λ. Then each P_a is strictly positive definite.*

Proof: For each a define

$$W_a = \{w_a \in V_a;\ P_a(w_a, w_a) = 0\}$$

and observe that for $w_a \in W_a$

$$0 = \lambda P_a(w_a, w_a) = \sum_b P_b(H_{ba}w_a, H_{ba}w_a).$$

Since the terms $P_a(H_{ba}w_a, H_{ba}w_a)$ on the right are all nonnegative, each term must be zero. Thus, $H_{ba}w_a \in W_b$. We conclude that $H_{ba}W_a \subseteq W_b$ for all a and b, which is to say that $(W_a)_a$ is an invariant subsystem. Irreducibility implies that either $W_a = 0$ or $W_a = V_a$ for all a. Since $P = (P_a)_a$ is nonzero it must be that $W_a = 0$ for all a, which means that each P_a is strictly positive definite. ∎

LEMMA 4.6: *If (V_a, H_{ba}) is a normalized irreducible system, then the subspace of vectors fixed by $\mathcal{T}$ is one-dimensional and contains a tuple $B = (B_a)_a$ of strictly positive definite forms.*

Proof: Because the system is normalized, Theorem 4.2 implies the existence of a nonzero tuple $B = (B_a)_a \in \mathcal{K}$ such that $\mathcal{T}B = B$. By Lemma 4.5, each B_a is strictly positive definite. Suppose that $P = (P_a) \in \mathcal{S}$ is any other tuple of symmetric sesquilinear forms satisfying $\mathcal{T}P = P$. If $P = 0$ there is nothing to prove. If P is negative semidefinite, replace it by $-P$. Then for some $a \in A$ and some $v \in V_a$ we will have that $P_a(v, v) > 0$. Consequently,

$$t_0 = \sup\{t \in \mathbf{R};\ B - tP \in \mathcal{K}\}$$

is not infinite. Since $\mathcal{K}$ is closed, $B - t_0P \in \mathcal{K}$. Since the set of strictly positive definite tuples is open, it cannot be that $B - t_0P$ is strictly positive definite. By Lemma 4.5 this implies that $B - t_0P = 0$. ∎

The proof of this next lemma is more or less copied from Vandergraft's paper.

LEMMA 4.7: *Let (V_a, H_{ba}) be a normalized irreducible system. If $P = (P_a)_a \in \mathcal{K}$ is a tuple of positive semidefinite sesquilinear forms satisfying $\mathcal{T}P = \lambda P$ for some positive $\lambda \neq 1$, then $P = 0$.*

Proof: As above, there is a positive definite tuple $B = (B_a)_a$ satisfying $\mathcal{T}B = B$. Assume that $P \in \mathcal{K}$ is nonzero. Since the spectral radius of $\mathcal{T}$ is $\rho = 1$, it must then be that $\lambda < 1$. According to Lemma 4.5, P is positive definite. Replacing P by tP for large positive t, we may assume that $P - B \in \mathcal{K}$. Then

$$-B = \lim_{n \to +\infty} (\lambda^n P - B) = \lim_{n \to +\infty} \mathcal{T}^n (P - B) \in \mathcal{K}.$$

That is to say, B is both (strictly) positive definite, and negative semidefinite, a contradiction. ∎

COROLLARY 4.8: *If the irreducible matrix system (V_a, H_{ba}) is not normalized, then no nonzero tuple $B = (B_a)_a \in \mathcal{K}$ is fixed by $\mathcal{T}$.*

Proof: Apply Lemma 4.7 to the normalized system $(V_a, \rho^{-1/2} H_{ba})$. ∎

THEOREM 4.9: *An irreducible matrix system (V_a, H_{ba}) has an extension to a system with inner products (V_a, H_{ba}, B_a) if and only if it is normalized. If such an extension exists, it is unique up to the multiplication of B by positive scalars t.*

Proof: Given the definition (Definition 3.5) of systems with inner products, the existence and uniqueness statements are in Lemma 4.6, while nonexistence is in Corollary 4.8. ∎

5. Irreducibility and inequivalence

Suppose we are given two systems with inner product, (V_a, H_{ba}, B_a) and $(V_a^\sharp, H_{ba}^\sharp, B_a^\sharp)$. From these we construct boundary representations π and $\pi^\sharp$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\sharp$. If we are given also a map of systems $(J_a)_a$, $J_a \colon V_a \to V_a^\sharp$, then we can define $\mathcal{J} \colon \mathcal{H} \to \mathcal{H}^\sharp$ by

$$\mathcal{J}(\mu[x, xa, v_a]) = \mu[x, xa, J_a v_a].$$

One checks that this is consistent with the relations

$$\mu[x, xa, v_a] = \sum_{b;\ ba \neq e} \mu[xa, xab, H_{ba} v_a]$$

and from this one deduces that $\mathcal{J}$ is well-defined on $\mathcal{H}^\infty$. Then one checks that it is a bounded map, with norm

$$\sup_a \Big(\sup_{v_a \in V_a;\ B_a(v_a,v_a)\le 1} B_a^\sharp(J_a v_a, J_a v_a)^{1/2} \Big)$$

and so can be extended to $\mathcal{H}$. Finally, one checks that it is a map of boundary representations, intertwining π and $\pi^\sharp$. Conversely

LEMMA 5.1: *Suppose that π and $\pi^\sharp$ are constructed from two systems with inner product (V_a, H_{ba}, B_a) and $(V_a^\sharp, H_{ba}^\sharp, B_a^\sharp)$. Suppose that (V_a, H_{ba}) is an irreducible system. Let $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}^\sharp$ be a continuous map intertwining the boundary representations π and $\pi^\sharp$. Then there exists a map of systems $(J_a)_a$ such that*

$$\mathcal{J}(\mu[x, xa, v_a]) = \mu[x, xa, J_a v_a].$$

The irreducibility hypothesis on (V_a, H_a) is actually necessary. Note also that the analogue of the Lemma which considers maps intertwining only the Γ-representations π_Γ and $\pi_\Gamma^\sharp$ is false. Indeed, one important objective of this series of papers is to identify those cases in which inequivalent irreducible systems (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$ give rise to equivalent Γ-representations π_Γ and $\pi_\Gamma^\sharp$.

Proof: If $\mathcal{J} = 0$, just choose $J_a = 0$. Otherwise, find some $v_b \in V_b$ so that

$$f^\sharp = \mathcal{J}(\mu[e, b, v_b]) \neq 0.$$

Fix $\epsilon > 0$. Since $\mathcal{H}^\infty$ is dense in $\mathcal{H}$, one can write, for N large enough,

$$\begin{aligned} f^\sharp &= f_N^\sharp + f_\epsilon^\sharp && \text{with } \|f_\epsilon^\sharp\| < \epsilon \|f_N^\sharp\|, \\ f_N^\sharp &= \sum_{|xa|=|x|+1=N+1} \mu[x, xa, w_{xa}^\sharp] && \text{with } w_{xa}^\sharp \in V_a^\sharp. \end{aligned}$$

Using the orthogonal decomposition $f_\epsilon^\sharp = \sum_{|xa|=|x|+1=N+1} \pi^\sharp(\mathbf{1}_{\Omega(xa)}) f_\epsilon^\sharp$, one deduces that there is at least one xa for which

$$\|\pi^\sharp(\mathbf{1}_{\Omega(xa)}) f_\epsilon^\sharp\| < \epsilon \|\mu[x, xa, w_{xa}^\sharp]\|.$$

In particular, $\mu[x, xa, w_{xa}^\sharp] \neq 0$, hence $\mu[x, xa, w_{xa}^\sharp] + \pi^\sharp(\mathbf{1}_{\Omega(xa)}) f_\epsilon^\sharp \neq 0$.

We work with this particular xa. One has $\pi(\mathbf{1}_{\Omega(xa)})\mu[e, b, v_b] = \mu[x, xa, w]$ for some $w \in V_a$. Because $\mathcal{J}$ intertwines π_Ω and $\pi_\Omega^\sharp$,

$$\mathcal{J}(\mu[x, xa, w]) = \pi^\sharp(\mathbf{1}_{\Omega(xa)}) \mathcal{J}(\mu[e, b, v_b]) = \mu[x, xa, w_{xa}^\sharp] + \pi^\sharp(\mathbf{1}_{\Omega(xa)}) f_\epsilon^\sharp.$$

In particular $w \neq 0$. Because $\mathcal{J}$ intertwines π_Γ and $\pi_\Gamma^\sharp$, one has

$$\mathcal{J}(\mu[e,a,w]) = \pi^\sharp(x^{-1})\mathcal{J}(\mu[x,xa,w]) = \mu[e,a,w^\sharp_{xa}] + \pi^\sharp(x^{-1})\pi^\sharp(\mathbf{1}_{\Omega(xa)})f^\sharp_\epsilon.$$

Now we normalize. Let $w_\epsilon = w/B_a(w,w)^{1/2}$, let $w^\sharp_\epsilon = w^\sharp_{xa}/B_a(w,w)^{1/2}$, and let $g^\sharp_\epsilon = \pi^\sharp(x^{-1})\pi^\sharp(\mathbf{1}_{\Omega(xa)})f^\sharp_\epsilon/B_a(w,w)^{1/2}$. Then

$$\mathcal{J}(\mu[e,a,w_\epsilon]) = \mu[e,a,w^\sharp_\epsilon] + g^\sharp_\epsilon \tag{5.1}$$

with $B_a(w_\epsilon,w_\epsilon) = 1$ and $\|g^\sharp_\epsilon\| < \epsilon\|\mu[e,a,w^\sharp_\epsilon]\|$. From these it follows that $B^\sharp_a(w^\sharp_\epsilon,w^\sharp_\epsilon) \le C_0 = \|\mathcal{J}\|^2/(1-\epsilon)^2$.

Clearly $a = a_\epsilon$ may depend on ϵ. Consider any sequence of ϵ's descending to 0. Passing to subsequences, we may assume that $a = a_{\epsilon_j}$ is constant, that $w_{\epsilon_j} \to w_0 \in V_a$ with $B_a(w_0,w_0) = 1$, and that $w^\sharp_{\epsilon_j} \to w^\sharp_0 \in V^\sharp_a$ with $B^\sharp_a(w^\sharp_0,w^\sharp_0) \le C_0$. Passing to the limit in (5.1) we obtain

$$\mathcal{J}(\mu[e,a,w_0]) = \mu[e,a,w^\sharp_0].$$

For all a, define

$$W_a = \{w_a \in V_a;\ \mathcal{J}(\mu[e,a,w_a]) = \mu[e,a,w^\sharp_a] \text{ for some } w^\sharp_a \in V^\sharp_a\}.$$

We have just shown that at least one of the W_a is nonzero. For $w_a \in W_a$ with $\mathcal{J}(\mu[e,a,w_a]) = \mu[e,a,w^\sharp_a]$ and for any $b \neq a^{-1}$,

$$\begin{aligned}\mathcal{J}(\mu[a,ab,H_{ba}w_a]) &= \mathcal{J}(\pi(\mathbf{1}_{\Omega(ab)})\mu[e,a,w_a]) = \pi^\sharp(\mathbf{1}_{\Omega(ab)})\mathcal{J}(\mu[e,a,w_a])\\ &= \pi^\sharp(\mathbf{1}_{\Omega(ab)})\mu[e,a,w^\sharp_a] = \mu[a,ab,H^\sharp_{ba}w^\sharp_a].\end{aligned}$$

Now translate by a^{-1} to see that

$$\mathcal{J}(\mu[e,b,H_{ba}w_a]) = \mu[e,b,H^\sharp_{ba}w^\sharp_b]. \tag{5.2}$$

Hence $(W_a)_a$ is an invariant subsystem of the irreducible system (V_a,H_{ba}). It follows that $W_a = V_a$ for all a. Thus, we can define $J_a\colon V_a \to V^\sharp_a$ according to

$$\mathcal{J}(\mu[e,a,v_a]) = \mu[e,a,J_a v_a]$$

and then deduce from (5.2) that $(J_a)_a$ is indeed a map of systems. ∎

Lemma 5.2: *If $(J_a)_a$ is a map from the irreducible system (V_a,H_{ba}) to itself then from some scalar λ we have $J_a = \lambda$ for all a.*

Proof: Choose some $b \in A$ for which $V_b \neq 0$. Then choose some eigenvalue λ of $J_b\colon V_b \to V_b$. For all a define $W_a = \ker(\lambda - J_a)$. If $w_a \in W_a$, then

$$J_b(H_{ba}w_a) = H_{ba}J_a(w_a) = \lambda H_{ba}w_a$$

which means that $H_{ba}w_a \in W_b$. Thus $(W_a)_a$ is a nonzero invariant subsystem of (V_a, H_{ab}). By irreducibility, it follows that $W_a = V_a$, completing the proof. ∎

The following theorem is now obvious.

THEOREM 5.3: *The boundary representation π constructed from a normalized irreducible system (V_a, H_{ba}) is itself irreducible. The boundary representations π and $\pi^\sharp$ constructed from two normalized irreducible systems (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$ are equivalent if and only if the two systems are equivalent.*

Proof: For the first part, let $\mathcal{H}_1$ be a closed π-invariant subspace of $\mathcal{H}$, and let $P\colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection with image $\mathcal{H}_1$. As per Observations 2.3, P is a map of boundary representations. According to Lemma 5.1, P is obtainable from a map of systems, $(J_a)_a$, and according to Lemma 5.2, that map of systems is actually given by a scalar λ. Hence $P = \lambda$, where clearly $\lambda = 0$ or $\lambda = 1$: the π-invariant subspace must be either 0 or all of $\mathcal{H}$.

Now consider two irreducible systems, (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$. If $(J_a)_a$, $J_a\colon V_a \to V_a^\sharp$ is an equivalence of systems, then B_a and $B_a^\sharp$, which are unique (up to positive constants), must actually correspond (up to a positive constant):

$$B_a^\sharp(J_a v_a, J_a w_a) = t_0 B_a(v_a, w_a)$$

for some $t_0 > 0$. As far as the equivalence class of π goes, the exact choice of $(B_a)_a$ makes no difference, so we may assume $t_0 = 1$. The boundary representations π and $\pi^\sharp$ constructed from the equivalent systems with inner product (V_a, H_{ba}, B_a) and $(V_a^\sharp, H_{ba}^\sharp, B_a^\sharp)$ are obviously equivalent; the obvious unitary intertwiner $\mathcal{J}$ is defined by $\mathcal{J}(\mu[e, a, v_a]) = \mu[e, a, J_a v_a]$.

Conversely, if $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}^\sharp$ is nonzero and intertwines the boundary representations π and $\pi^\sharp$, then according to Lemma 5.1 it is obtainable from a nonzero map of systems $(J_a)_a$, and according to Remark 3.4 that map is an equivalence of systems. ∎

The following Corollary is analogous to Theorem 5.2 of Kuhn–Steger, 1996 [K–S2]. That paper treats the scalar case, $\dim V_a = 1$. In these papers, as in that one, the result is crucial for proving irreducibility and inequivalence for Γ-representations π_Γ arising from irreducible systems (V_a, H_{ba}). Although the introduction of Hilbert spaces and boundary representations is not necessary either for the statement or the proof of the Corollary, the most natural way to prove its second part is by referring to Theorem 5.3 above.

COROLLARY 5.4: *Given two normalized irreducible systems* (V_a, H_{ba}) *and* $(V_a^\sharp, H_{ba}^\sharp)$, *let* $\mathcal{V} = \oplus_a V_a' \otimes (V_a^\sharp)^*$ *and define* $T\colon \mathcal{V} \to \mathcal{V}$ *by*

$$(TC)_a = \sum_a T_{ab} C_b, \quad T_{ab} = (H_{ba})' \otimes (H_{ba}^\sharp)^*.$$

Then

(1) *The spectral radius* $\rho(T)$ *is less than or equal to* 1.
(2) 1 *is an eigenvalue of* T *if and only if the two systems are equivalent.*

Proof: For part (2), first use Theorem 4.9 to find $(B_a)_a$ and $(B_a^\sharp)_a$, and then use the construction of Section 3 to construct boundary representations π and $\pi^\sharp$ based on the systems with inner product (V_a, H_{ba}, B_a) and $(V_a^\sharp, H_{ba}^\sharp, B_a^\sharp)$. Suppose $C = (C_a)_a$ is a nonzero fixed vector for T. View $C_a \in V_a' \otimes (V_a^\sharp)^*$ as a sesquilinear form $C_a(v_a, v_a^\sharp)$ defined on $V_a \times V_a^\sharp$. Then

$$\begin{aligned}\sum_b C_b(H_{ba}v_a, H_{ba}^\sharp v_a^\sharp) &= \sum_b (((H_{ba})' \otimes (H_{ba}^\sharp)^*)C_b)(v_a, v_a^\sharp)\\ &= (TC)_a(v_a, v_a^\sharp) = C_a(v_a, v_a^\sharp).\end{aligned} \tag{5.3}$$

For $f \in \mathcal{H}^\infty$ and $f^\sharp \in {\mathcal{H}^\sharp}^\infty$, both satisfying (3.2) for a certain value of N, define

$$\mathcal{C}(f, f^\sharp) = \sum_{|xa|=|x|+1=N+1} C_a(f(xa), f^\sharp(xa)).$$

Using (5.3), one sees that this is well defined, independent of N. Moreover, $|\mathcal{C}(f, f^\sharp)| \le k\|f\|\,\|f^\sharp\|$ where

$$k = \sup\{|C_a(v_a, v_a^\sharp)|;\ a \in A, B_a(v_a, v_a) \le 1, B_a^\sharp(v_a^\sharp, v_a^\sharp) \le 1\},$$

implying that one may extend $\mathcal{C}$ to a sesquilinear form on $\mathcal{H} \times \mathcal{H}^\sharp$.

For $|xa| = |x| + 1$ and $|y| < |x|$, note that

$$\begin{aligned}\mathcal{C}(\pi(y)\mu[x, xa, v_a], \pi^\sharp(y)\mu[x, xa, v_a^\sharp]) &= C_a(v_a, v_a^\sharp)\\ &= \mathcal{C}(\mu[x, xa, v_a], \mu[x, xa, v_a^\sharp]).\end{aligned}$$

Then use linearity, orthogonality and continuity to check that

$$\mathcal{C}(\pi(y)f, \pi^\sharp(y)f^\sharp) = \mathcal{C}(f, f^\sharp) \tag{5.4}$$

for arbitrary $y \in \Gamma$, $f \in \mathcal{H}$, and $f^\sharp \in \mathcal{H}^\sharp$. Next, note that

$$\mathcal{C}(\pi(\mathbf{1}_{\Omega(x)})f, f^\sharp) = \mathcal{C}(f, \pi^\sharp(\mathbf{1}_{\Omega(x)})f^\sharp)$$

for $f \in \mathcal{H}^\infty$ and $f^\sharp \in \mathcal{H}^{\sharp\infty}$. Then use linearity and continuity to check that

$$\mathcal{C}(\pi(F)f, f^\sharp) = \mathcal{C}(f, \pi^\sharp(\bar{F})f^\sharp) \tag{5.5}$$

for arbitrary $F \in C(\Omega)$, $f \in \mathcal{H}$, and $f^\sharp \in \mathcal{H}^\sharp$.

Define $\mathcal{J}\colon \mathcal{H}^\sharp \to \mathcal{H}$ according to $\mathcal{C}(f, f^\sharp) = \langle f, \mathcal{J} f^\sharp\rangle_{\mathcal{H}}$. From equations (5.4) and (5.5), it follows that $\mathcal{J}$ is a nonzero intertwiner between π and $\pi^\sharp$. Then according to Theorem 5.3 the two systems (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$ are equivalent, as desired.

Conversely, if the two systems (V_a, H_{ba}) and $(V_a^\sharp, H_{ba}^\sharp)$ are equivalent, then one may as well assume they are identical. A nonzero fixed vector for T is then given by $(B_a)_a$.

For part (1), suppose that $C = (C_a)$ satisfies $TC = \lambda C$. Then

$$\begin{aligned}\sum_b C_b(H_{ba}v_a, H_{ba}^\sharp v_a^\sharp) &= \sum_b (((H_{ba})' \otimes (H_{ba}^\sharp)^*)C_b)(v_a, v_a^\sharp)\\ &= (TC)_a(v_a, v_a^\sharp) = \lambda C_a(v_a, v_a^\sharp).\end{aligned}$$

Define k as before:

$$k = \sup\{|C_a(v_a, v_a^\sharp)|;\ a \in A, B_a(v_a, v_a) \le 1, B_a^\sharp(v_a^\sharp, v_a^\sharp) \le 1\}.$$

Then for any $v_a \in V_a$, $v_a^\sharp \in V_a^\sharp$

$$\begin{aligned}|\lambda C_a(v_a, v_a^\sharp)| &= \left|\sum_b C_b(H_{ba}v_a, H_{ba}^\sharp v_a^\sharp)\right|\\ &\le k\sum_b B_b(H_{ba}v_a, H_{ba}v_a)^{1/2} B_b^\sharp(H_{ba}^\sharp v_a^\sharp, H_{ba}^\sharp v_a^\sharp)^{1/2}\\ &\le k\left(\sum_b B_b(H_{ba}v_a, H_{ba}v_a)\right)^{1/2}\left(\sum_b B_b^\sharp(H_{ba}^\sharp v_a^\sharp, H_{ba}^\sharp v_a^\sharp)\right)^{1/2}\\ &= kB_a(v_a, v_a)^{1/2} B_a^\sharp(v_a^\sharp, v_a^\sharp)^{1/2}\end{aligned}$$

from which we deduce $|\lambda k| \le k$. If $|\lambda| > 1$, it must be that $k = 0$, hence that $C = 0$. This means that any eigenvalue λ of T satisfies $|\lambda| \le 1$. ∎

6. Examples

We show here how to obtain some of the representations listed in the introduction from our construction. Most of the details are left to the reader. With one exception the multiplicative functions here are all going to be *scalar-valued*. That is, we will almost always have $V_a = \mathbf{C}$ or $V_a = 0$.

6.1. THE CASE $\Gamma = \mathbf{Z}$. Assume that there is only one generator, called c. Of the four maps H_{ab}, only H_{cc} and $H_{c^{-1}c^{-1}}$ can be nonzero. The irreducibility condition forces one of the two vector spaces V_c and $V_{c^{-1}}$ to be zero, and the other to have dimension 1. Assume $V_{c^{-1}} = 0$. The system will be normalized if and only if $H_{cc} = e^{-i\theta}$ for some real θ. Taking account of the equivalence relation in its definition, one sees that $\mathcal{H}^\infty$ is one-dimensional with basis vector $\mu[e, c, e^{-i\theta}]$:

$$\mu[e,c,e^{-i\theta}](c^n) = \begin{cases} 0 & \text{if } n \le 0 \\ e^{-in\theta} & \text{if } n > 0 \end{cases}$$

where $\pi(c)\mu[e,c,e^{-i\theta}] = e^{i\theta}\mu[e,c,e^{-i\theta}]$.

For the case $\Gamma = \mathbf{Z}$, Ω consists of exactly two points, $c^{+\infty}$ and $c^{-\infty}$, and the boundary representation described above is supported entirely on $c^{+\infty}$. The same Γ-representation is also obtainable by letting $V_c = 0$, $V_{c^{-1}} = \mathbf{C}$ and $H_{c^{-1}c^{-1}} = e^{i\theta}$. In that case, the boundary representation is supported entirely on $c^{-\infty}$. This is an example of *duplicity*, as described at the end of Section 2.

6.2. YOSHIZAWA'S REPRESENTATIONS π_Y. Yoshizawa constructed these in 1951 [Yos] by inducing unitary characters χ of $\mathbf{Z}$ up to Γ. Specifically, identify $\mathbf{Z}$ with the subgroup generated by $c \in A$. If $\chi(c^n) = e^{in\theta}$, let $v_0\colon \Gamma \to \mathbf{C}$ be the function in the representation space $\mathcal{H}_Y$ of $\mathrm{Ind}_{\mathbf{Z}}^{\Gamma}\chi$ defined by

$$v_0(x) = \begin{cases} e^{-in\theta} & \text{if } x = c^n, \\ 0 & \text{if } x \text{ is not of the form } c^n. \end{cases}$$

Set $V_c = \mathbf{C}$ and $V_a = 0$ for $a \neq c$. Set $H_{cc} = e^{-i\theta}$ and $H_{ba} = 0$ otherwise. This gives a normalized irreducible system and one can choose $B_c(v,v) = |v|^2$. In the representation space $\mathcal{H}^\infty$ constructed in Section 3 one has the vector

$$\mu[e,c,e^{-i\theta}](x) = \begin{cases} e^{-in\theta} & \text{if } x = c^n \text{ with } n > 0, \\ 0 & \text{if } x \text{ is not of the form } c^n \text{ with } n > 0. \end{cases}$$

There exists an equivalence $\mathcal{J}\colon \mathcal{H}_Y \to \mathcal{H}$ of Γ-representations determined by $\mathcal{J}(v_0) = \mu[e,c,e^{-i\theta}]$. Indeed, it is easy to check that the matrix coefficients of v_0 and $\mu[e,c,e^{-i\theta}]$ coincide and that $\mu[e,c,e^{-i\theta}]$ is cyclic for π_Γ.

As in the previous example the same representation of Γ can be obtained via the alternate choices $V_{c^{-1}} = \mathbf{C}$, $V_b = 0$ otherwise, $H_{c^{-1}c^{-1}} = e^{i\theta}$, and $H_{ba} = 0$ otherwise.

6.3. THE SPHERICAL PRINCIPAL SERIES. This series of representations was constructed by Figà-Talamanca and Picardello in 1982 [F-T–P1] and 1983

[F-T–P2]. Each of them acts on $L^2(\Omega, d\nu)$ where $d\nu$ is the *isotropic* probability measure on Ω which for every n assigns equal measure to all of the sets $\{\Omega(x)\}_{|x|=n}$. Letting $\#(A) = q+1$, one finds $\nu(\Omega(x)) = \frac{q}{q+1}q^{-|x|}$.

For any real s, let π_s be the corresponding spherical principal series representation. To obtain from the construction described in this paper a boundary representation π such that π_Γ is equivalent to π_s, choose $V_a = \mathbf{C}$ for all a and $H_{ba} = q^{-1/2+is}$ for all a, b with $ab \neq e$. It is convenient to choose $B_a(v,v) = \frac{1}{q+1}|v|^2$ for each a. Define a unitary operator $\mathcal{J}\colon \mathcal{H}^\infty \to L^2(\Omega, d\nu)$ by letting

$$\mathcal{J}(\mu[x, xa, 1]) = (q^{(\frac{1}{2}-is)|x|})\mathbf{1}_{\Omega(xa)}$$

whenever $|xa| = |x|+1$. Then

$$\langle \mu[x, xa, 1], \mu[x, xa, 1]\rangle = \frac{1}{q+1} = \langle \mathcal{J}(\mu[x, xa, 1]), \mathcal{J}(\mu[x, xa, 1])\rangle.$$

Using this, one can check that $\mathcal{J}$ is isometric, and that it extends to a unitary map $\mathcal{J}\colon \mathcal{H} \to L^2(\Omega, d\nu)$. Using the definition of the spherical principal series, one can also check that $\mathcal{J}$ is an equivalence of Γ-representations.

The alternative choice $H_{ba} = q^{-1/2-is}$ gives rise to π_{-s}. One knows that π_s and π_{-s} are equivalent as Γ-representations, while from Theorem 5.3, one can deduce that they are not equivalent as boundary representations (representations of $\Gamma \ltimes_\lambda C(\Omega)$) unless $q^{is} = \pm 1$. Actually, π_s and π_{-s} would be equivalent as boundary representations if and only if the intertwiner between π_s and π_{-s} was of the form $f(\omega) \mapsto U(\omega)f(\omega)$ for some complex function $U(\omega)$, $|U(\omega)| = 1$, whereas the true intertwiner is a singular integral operator.

6.4. Tensor products with the spherical principal series. These are discussed in the 1996 paper with Pensavalle [P–S]. Let $\rho\colon \Gamma \to \mathcal{U}(\mathcal{V})$ be an irreducible finite-dimensional unitary representation of Γ. Let π_s be as above. To avoid a certain interesting complication, suppose that $q^{is} \neq -1$. Then $\pi_s \otimes \rho$ is irreducible. One identifies the representation space $L^2(\Omega, d\nu) \otimes \mathcal{V}$ with $L^2(\Omega, d\nu, \mathcal{V})$.

To get $\pi_s \otimes \rho$ from a matrix system, set $V_a = \mathcal{V}$ for all a and set $H_{ba} = q^{-1/2+is}\rho^{-1}(b)$ for all a, b with $ab \neq e$. Define $\mathcal{J}\colon \mathcal{H}^\infty \to L^2(\Omega, d\nu, \mathcal{V})$ by

$$\mathcal{J}(\mu[x, xa, v]) = (q^{(1/2-is)|x|})\mathbf{1}_{\Omega(xa)}\rho(xa)v$$

whenever $|xa| = |x|+1$. One can check that this is an intertwiner of Γ-representations.

Here we have passed from an original matrix system (V_a, H_{ba}) to a new matrix system $(V_a \otimes \mathcal{V}, H_{ba} \otimes \rho^{-1}(b))$. It is not hard to check that this always corresponds to passing from π to $\pi \otimes \rho$. In the specific case described here the new matrix system is still irreducible, but this does not hold in general.

6.5. Paschke's representations. These representations are explained in Paschke, 2001 [P1]. Let $n = \#A_+ = \#A_- = \#A/2$ and choose $h, p > 0$ satisfying

$$nh^2 + (n-1)p^2 = 1.$$

Let $\delta_x \in \ell^1(\Gamma)$ denote the Kronecker delta at the point x. Define $u_\pm \in \ell^1(\Gamma)$ by

$$u_\pm = \left(\sum_{a \in A_\pm} \delta_a \right) - nh\delta_e.$$

Paschke's representation, π_+, is a tempered representation of Γ, defined by giving the matrix coefficient of a particular cyclic vector Δ_1^+. This cyclic vector satisfies $\pi_+(u_+)\Delta_1^+ = 0$. We will also need π_-, defined in analogy with π_+, but exchanging the roles of A_+ and A_-.

Paschke conjectures that any irreducible tempered representation ρ of Γ such that $\rho(u_+)$ has a nontrivial kernel is equivalent to π_+. This is part of a more general conjecture where in place of u_+ one considers a general finitely supported function in $\ell^1(\Gamma)$. See Paschke's papers of 2001 [P1] and 2002 [P2] for the details.

Our construction will give a boundary representation π which, upon restriction to Γ, is equivalent to $\pi_+ \oplus \pi_-$. This is an example of *oddity*, as described at the end of Section 2. Choose $V_a = \mathbf{C}$ for all a and

$$H_{ba} = \begin{cases} h & \text{if } a, b \in A_+ \text{ or if } a, b \in A_-, \\ p & \text{if } a \in A_- \text{ and } b \in A_+,\ ab \neq e, \\ -p & \text{if } a \in A_+ \text{ and } b \in A_-,\ ab \neq e. \end{cases} \tag{6.1}$$

Set also $B_a(v, v) = |v|^2$ for all a. Define

$$f_\pm = \sum_{a \in A_\pm} \mu[e, a, h] + \sum_{a \in A_\mp} \mu[e, a, \mp p].$$

It is an easy exercise to verify that $\pi(u_\pm) f_\pm = 0$.

Somewhat harder is the calculation of the matrix coefficients for $f_\pm$. If $a_1^{\epsilon_1} \dots a_m^{\epsilon_m}$ is a reduced word with $a_j \in A_+$ and $\epsilon_j = \pm 1$, then define $|a_1^{\epsilon_1} \dots a_m^{\epsilon_m}| = m$, as usual, and

$$|a_1^{\epsilon_1} \dots a_m^{\epsilon_m}|_{-+} = \#\{j;\ \epsilon_j = -1, \epsilon_{j+1} = +1\},$$
$$|a_1^{\epsilon_1} \dots a_m^{\epsilon_m}|_{+-} = \#\{j;\ \epsilon_j = +1, \epsilon_{j+1} = -1\}.$$

One can calculate

$$\langle \pi(x)f_+, f_- \rangle = 0, \tag{6.2}$$

$$\langle \pi(x)f_+, f_+ \rangle = h^{|x|}\Big(-\frac{p^2}{h^2}\Big)^{|x|_{-+}}(1+p^2), \tag{6.3}$$

$$\langle \pi(x)f_-, f_- \rangle = h^{|x|}\Big(-\frac{p^2}{h^2}\Big)^{|x|_{+-}}(1+p^2). \tag{6.4}$$

Formula (6.2) shows that the subrepresentations of $\mathcal{H}$ generated by f_+ and f_- are mutually orthogonal. It is easy to check that their direct sum gives all of $\mathcal{H}$. The formula (6.3) for $\langle \pi(x)f_+, f_+\rangle$ matches Paschke's formula for $\langle \pi_+(x)\Delta_1^+, \Delta_1^+\rangle$ (at the beginning of Section 2 of [P1], where our h corresponds to Pascke's a). Hence the subrepresentation generated by f_+ is equivalent to π_+. Likewise, the subrepresentation generated by f_- is equivalent to π_-.

6.6. Endpoint Paschke representations. Paschke also defines his representation π_+ for the endpoint case:

$$h = n^{-1/2}, \quad p = 0.$$

For this endpoint case, equation (6.1) gives a *reducible* matrix system, which is easily seen to be the direct sum of two irreducible invariant subsystems. One of these subsytems is given by taking $V_a = \mathbf{C}$ for $a \in A_+$, $V_a = 0$ for $a \in A_-$, and $H_{ba} = h$ for $a, b \in A_+$. As above, the function in $\mathcal{H}^\infty$ corresponding to Δ_1^+ is $f_+ = \sum_{a\in A_+} \mu[e,a,h]$ and one may check that $\langle \pi(x)f_+, f_+\rangle = \langle \pi_+(x)\Delta_1^+, \Delta_1^+\rangle$.

In this endpoint case, there is only one matrix system (up to equivalence) for which π_Γ, as constructed here, is equivalent to π_+. Cases like these, which are neither odd nor duplicitous, are called **monotonous**. From this particular example, one would imagine that monotony is rather special, but actually it is the generic case when one looks at all matrix systems, or even at all scalar valued matrix systems. See our 1996 and 2001 papers [K–S2] and [K–S3].

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